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# Equations of General Relativistic Radiation Hydrodynamics from Tensor Formalism

Myeong-Gu Park[1]*
[1] *Kyungpook National University, Daegu 702-701, Korea*



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**ABSTRACT**

Radiation interacts with matter via exchange of energy and momentum. When matter is moving with a relativistic velocity or when the background spacetime is strongly curved, rigorous relativistic treatment of hydrodynamics and radiative transfer is required. Here, we derive fully general relativistic radiation hydrodynamic equations from a covariant tensor formalism. The equations can be applied to any three-dimensional problems and are rather starightforward to understand compared to the comoving frame-based equations. Current approach is applicable to any spacetime or coordinates, but in this work we specifically choose the Schwarzschild spacetime to show explicitly how the hydrodynamic and the radiation moment equations are derived. Some important aspects of relativistic radiation hydrodynamics and the difficulty with the radiation moment formalism are discussed as well.

**Key words:** accretion – hydrodynamics – radiative transfer – relativity

## 1 INTRODUCTION

Radiation interacts with matter. The dynamics and energy contents of matter are influenced by surrounding radiation while radiation itself is concurrently determined by surrounding matter. So one needs to solve the hydrodynamic and the radiative transfer equations simultaneously and self-consistently.

When matter is moving with a relativistic velocity, various relativistic effects should be taken into account: Doppler shift, aberration, time dilation, relativistic beaming, and so forth. These make physical quantities observer-dependent: one needs to specify carefully the observer from whose point of view physical quantities are measured. The same is true for matter and radiation in curved spacetime. Gravitational redshift, time dilation, loss cone effect, and so forth should be taken into account. To consistently deal with all these relativistic effects, one needs to build the radiation hydrodynamics within a fully relativistic framework.

Thomas (1930) formulated special relativistic theory of radiative transfer to deal with radiative viscosity in a differentially moving media. Lindquist (1966) subsequently developed a covariant form of general relativistic photon transport equation: a partial differential equation in time, space, angle, and frequency. A straightforward solution to such a fully angle- and frequency-dependent radiative transfer equation would be a direct integration of the multi-dimensional, partial differential equation. Although this can be done in principle by a finite difference method (Hauschildt & Wehrse 1991; Liebendörfer et al. 2004), characteristic methods which reduce the transfer equation to a single ordinary differential equation along characteristic rays are more practical and successfully applied to special relativistic (Mihalas 1980) and general relativistic problems (Schmid-Burgk 1978; Schinder 1988; Schinder & Bludman 1989; Zane et al. 1996). However, even these characteristic methods have been limited so far to spherically symmetric and steady cases. And if one is interested in the dynamics of the gas flow rather than in the details of the emergent spectrum, simpler description of the radiation field may suffice.

Lindquist (1966) also derived relativistic radiation moment equations from the transport equation. In the radiation moment formalism, a finite number of angular moments, generated from the specific intensity, are used to describe the radiation field. Thorne (1981) generalized the moment formalism to projected symmetric trace-free (PSTF) tensors to derive the relativistic version of moment equations up to an arbitrary order. PSTF formalism has been successfully applied to a variety of spherically symmetric problems (Flammang 1982, 1984; Turolla & Nobili 1988; Nobili, Turolla, & Zampieri 1991, 1993; Zampieri, Turolla, & Treves 1993).

Almost all aforementioned formalisms are based on the comoving frame: matter and radiation quantities and their directional derivatives are defined and described within the comoving frame. It is most convenient to define various matter and radiation quantities and describe their interactions in the comoving frame, a frame that moves along with mat-

* E-mail: mgp@knu.ac.kr

ter. But, the directional derivatives in the comoving frame are rather awkward because the velocity of matter varies in space and in time.

One can also present the radiation hydrodynamic equations in Eulerian framework: derivatives appear in much simpler form in the Eulerian frame. The most straightforward Eulerian frame is a frame fixed with respect to the central object, which we call the fixed frame. The methodology of using comoving radiation quantities within fixed-frame coordinates was first employed by Mihalas (1980), which we will call the mixed-frame formalism. Park (1993) also started from covariant tensor equations for energy-momentum to derive the mixed-frame radiation hydrodynamic equations for spherically symmetric systems: matter and radiation quantities and their derivatives were in covariant forms under fixed-frame coordinates while the interactions between radiation and matter were described by the comoving quantities. This approach makes the equations easier to interpret and apply to astrophysical problems. Recently, special relativistic radiation hydrodynamic equations for three-dimensional problems were derived by the same approach (Park 2004). In this work, we extend this mixed-frame formalism to arbitrary spacetime or coordinates. We show specifically how the three-dimensional, general relativistic radiation hydrodynamic equations are derived in the Schwarzschild geometry. We also discuss some aspects of relativistic radiation hydrodynamics as well as the difficulty in closing the radiation moment equations. We put $c = 1$ in most equations except in a few cases where $c$ is explicitly written for clarity.

## 2 TENSOR EQUATIONS

### 2.1 Matter

The energy-momentum tensor of an ideal gas is

$$T^{\alpha\beta} \equiv \omega_g U^\alpha U^\beta + P_g g^{\alpha\beta}, \tag{1}$$

where $U^\alpha$ is the four velocity of the gas and $\omega_g \equiv \varepsilon_g + P_g$ the gas enthalpy per unit proper volume which is the sum of the gas energy density $\varepsilon_g$ and the gas pressure $P_g$. The functional dependence of the gas enthalpy $\omega_g$ on the gas temperature $T$ can become complex, especially in the transrelativistic regime (Service 1986).

### 2.2 Radiation

The energy and momentum of the radiation field are represented by the radiation stress tensor that consists of zeroth, first, and second moments in angle of the radiation field,

$$R^{\alpha\beta} = \iint I(\mathbf{n}, \nu) n^\alpha n^\beta d\nu d\Omega, \tag{2}$$

where $I(x^\alpha; \mathbf{n}, \nu)$ is the specific intensity of photons moving in direction $\mathbf{n}$ on a unit sphere of projected tangent space with the frequency $\nu$ measured by a fiducial observer, $p^\alpha$ the four-momentum of photons, and $n^\alpha \equiv p^\alpha/h\nu$. Since $I_\nu/\nu^3$ which is proportional to the photon distribution function and $\nu d\nu d\Omega$ are frame-independent scalars, $R^{\alpha\beta}$ is a contravariant tensor.

### 2.3 Radiation Hydrodynamic Equations in Covariant Form

Since mass and energy are equivalent in relativity, it is the particle number density, rather than the mass density, that is conserved in relativistic hydrodynamics. The continuity equation, therefore, should represent the conservation of the number density,

$$(nU^\alpha)_{;\alpha} = 0. \tag{3}$$

In the absence of any external force other than the radiative interaction, the total energy-momentum tensor of gas plus radiation is conserved,

$$\left(T^{\alpha\beta} + R^{\alpha\beta}\right)_{;\beta} = 0. \tag{4}$$

One can define the radiation four-force density to specifically describe the energy and momentum transferred from radiation to gas (Mihalas & Mihalas 1984),

$$G^\alpha \equiv \frac{1}{c}\int d\nu \int d\Omega[\chi I(\mathbf{n}, \nu) - \eta] n^\alpha, \tag{5}$$

where $\chi$ is the opacity per unit proper length and $\eta$ the emissivity per proper unit volume. The combinations $\eta/\nu^2$ and $\nu\chi$ are again frame-independent scalars.

This radiation four-force density is equal to the divergence of the energy-momentum tensor for matter,

$$T^{\alpha\beta}{}_{;\beta} = G^\alpha, \tag{6}$$

and to the minus the divergence of the radiation stress tensor,

$$R^{\alpha\beta}{}_{;\beta} = -G^\alpha. \tag{7}$$

When micro-physical processes for the interactions between radiation and matter are known, equation (4) can be put into two separate equations (6) and (7) and solved.

## 3 SCHWARZSCHILD SPACETIME

The above tensor equations (3)–(7) can be explicitly written out in any spacetime or coordinates . In this work, we specifically choose the Schwarzschild spacetime to show explicitly how to derive more familiar hydrodynamic and radiation moment equations and various transformation relations among radiation moments. Much simpler, spherically symmetric case has been shown in Park (1993).

### 3.1 Schwarzschild geometry

The most familiar form of the Schwarzschild metric is

$$\begin{aligned} d\tau^2 &= -g_{\alpha\beta} dx^\alpha dx^\beta \\ &= \Gamma^2 dt^2 - \frac{dr^2}{\Gamma^2} - r^2(d\theta^2 + \sin^2\theta d\phi^2), \end{aligned} \tag{8}$$

where $M$ is the mass of the central object, $m \equiv GM/c^2$, and

$$\Gamma \equiv \left(1 - \frac{2m}{r}\right)^{1/2} \tag{9}$$

is the lapse function.

The four-velocity of the gas, defined as

$$U^\alpha \equiv \frac{dx^\alpha}{d\tau}, \tag{10}$$

satisfies the normalization condition

$$U_\alpha U^\alpha = -1, \tag{11}$$

from which one can define the energy parameter

$$\begin{aligned} y &\equiv -U_t \\ &= \left[\Gamma^2 + (U^r)^2 + \Gamma^2 (rU^\theta)^2 + \Gamma^2 (r\sin\theta U^\phi)^2\right]^{1/2}. \end{aligned} \tag{12}$$

### 3.2 Tetrads

It is not trivial to define physical quantities for matter or radiation in curved spacetime. However, one can always set up some tetrad, and since a tetrad frame is a locally inertial frame, physical quantities can be straightforwardly defined as in flat spacetime. Among various tetrads, fixed and comoving tetrads are the most relevant ones. A fixed tetrad is an orthonormal tetrad fixed with respect to the coordinates and has base $\mathbf{e}_{\hat{i}} = \partial/\partial x^{\hat{i}}$ that can be expressed in terms of coordinate base $\partial/\partial x^i$ as

$$\begin{aligned} \frac{\partial}{\partial \hat{t}} &= \frac{1}{\Gamma}\frac{\partial}{\partial t} \\ \frac{\partial}{\partial \hat{r}} &= \Gamma \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \hat{\theta}} &= \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial \hat{\phi}} &= \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}. \end{aligned} \tag{13}$$

Physical quantities defined in the fixed tetrad are those measured by a fiducial observer who is fixed with respect to the coordinates. This observer sees matter moving with the proper velocity $\boldsymbol{v}$, which is related to the four-velocity as

$$v^i = \frac{U^{\hat{i}}}{U^{\hat{t}}} = \frac{U_\alpha e_{\hat{i}}{}^\alpha}{-U_\alpha e_{\hat{t}}{}^\alpha}, \tag{14}$$

where $U^{\hat{i}}$ and $U^{\hat{t}}$ are the spatial and time parts of the fixed tetrad components of the four-velocity and $e_{\hat{i}}{}^\alpha$ is the $\alpha$-th component of the tetrad base $\mathbf{e}_{\hat{i}}$. Specifically,

$$\begin{aligned} v^r &= \frac{1}{y}U^r, \\ v^\theta &= \frac{\Gamma}{y} r U^\theta, \\ v^\phi &= \frac{\Gamma}{y} r\sin\theta U^\phi. \end{aligned} \tag{15}$$

Since $\boldsymbol{v}$ is a three vector defined in the tetrad, $v_i = v^i$. We also define the Lorentz factor $\gamma$ for $\boldsymbol{v}$ as

$$\gamma \equiv (1 - v^2)^{-1/2} = \frac{y}{\Gamma}, \tag{16}$$

where $v^2 = \boldsymbol{v}\cdot\boldsymbol{v} = v_i v^i = v_r^2 + v_\theta^2 + v_\phi^2$. The value of $v^2$ at the horizon is always 1 regardless of $U^i$: a fiducial observer fixed at the horizon always sees matter radially falling in with velocity $c$.

A comoving tetrad moves with velocity $v^i$ relative to the fixed tetrad and therefore is related to the fixed tetrad by the Lorentz tranformation,

$$\frac{\partial}{\partial x^{\hat{\alpha}}_{co}} = \Lambda^{\hat{\beta}}{}_{\hat{\alpha}}(\boldsymbol{v}) \frac{\partial}{\partial x^{\hat{\beta}}}, \tag{17}$$

where $\hat{\alpha}$ and $\hat{\beta}$ denote tetrad's bases. The components of Lorentz transformation $\Lambda^{\hat{\alpha}}{}_{\hat{\beta}}(\boldsymbol{v})$ are

$$\begin{aligned} \Lambda^{\hat{t}}{}_{\hat{t}} &= \gamma \\ \Lambda^{\hat{i}}{}_{\hat{t}} &= \gamma v^i \\ \Lambda^{\hat{t}}{}_{\hat{j}} &= \gamma v_j \\ \Lambda^{\hat{i}}{}_{\hat{j}} &= \delta^i{}_j + v^i v_j \frac{\gamma - 1}{v^2}. \end{aligned} \tag{18}$$

We can also express the comoving tetrad in terms of coordinate base, which is useful when converting the tetrad components to and from covariant ones:

$$\begin{aligned} \frac{\partial}{\partial \hat{t}_{co}} &= \frac{\gamma}{\Gamma}\frac{\partial}{\partial t} + \gamma\Gamma v_r \frac{\partial}{\partial r} + \gamma v_\theta \frac{1}{r}\frac{\partial}{\partial\theta} + \gamma v_\phi \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \\ \frac{\partial}{\partial \hat{r}_{co}} &= \frac{\gamma}{\Gamma} v_r \frac{\partial}{\partial t} + \Gamma\left[1 + (\gamma-1)\frac{v_r^2}{v^2}\right]\frac{\partial}{\partial r} \\ &+ (\gamma-1)\frac{v_r v_\theta}{v^2}\frac{1}{r}\frac{\partial}{\partial\theta} + (\gamma-1)\frac{v_r v_\phi}{v^2}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}, \\ \frac{\partial}{\partial \hat{\theta}_{co}} &= \frac{\gamma}{\Gamma} v_\theta \frac{\partial}{\partial t} + \Gamma(\gamma-1)\frac{v_r v_\theta}{v^2}\frac{\partial}{\partial r} \\ &+ \left[1 + (\gamma-1)\frac{v_\theta^2}{v^2}\right]\frac{1}{r}\frac{\partial}{\partial\theta}, \\ &+ (\gamma-1)\frac{v_\theta v_\phi}{v^2}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial \hat{\phi}_{co}} &= \frac{\gamma}{\Gamma} v_\phi \frac{\partial}{\partial t} + \Gamma(\gamma-1)\frac{v_r v_\phi}{v^2}\frac{\partial}{\partial r} \\ &+ (\gamma-1)\frac{v_\theta v_\phi}{v^2}\frac{1}{r}\frac{\partial}{\partial\theta} \\ &+ \left[1 + (\gamma-1)\frac{v_\phi^2}{v^2}\right]\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}. \end{aligned} \tag{19}$$

Similarly, applying the inverse Lorentz transformation $\Lambda^{\hat{\alpha}}{}_{\hat{\beta}}(-\boldsymbol{v})$ yields the inverse transformation from the comoving tetrad to the coordinate base:

$$\begin{aligned} \frac{1}{\Gamma}\frac{\partial}{\partial t} &= \gamma\frac{\partial}{\partial \hat{t}_{co}} - \gamma v_r \frac{\partial}{\partial \hat{r}_{co}} - \gamma v_\theta \frac{\partial}{\partial \hat{\theta}_{co}} - \gamma v_\phi \frac{\partial}{\partial \hat{\phi}_{co}}, \\ \Gamma\frac{\partial}{\partial r} &= -\gamma v_r \frac{\partial}{\partial \hat{t}_{co}} + \left[1 + (\gamma-1)\frac{v_r^2}{v^2}\right]\frac{\partial}{\partial \hat{r}_{co}} \\ &+ (\gamma-1)\frac{v_r v_\theta}{v^2}\frac{\partial}{\partial \hat{\theta}_{co}} + (\gamma-1)\frac{v_r v_\phi}{v^2}\frac{\partial}{\partial \hat{\phi}_{co}}, \\ \frac{1}{r}\frac{\partial}{\partial\theta} &= -\gamma v_\theta \frac{\partial}{\partial \hat{t}_{co}} + (\gamma-1)\frac{v_\theta v_r}{v^2}\frac{\partial}{\partial \hat{r}_{co}} \\ &+ \left[1 + (\gamma-1)\frac{v_\theta^2}{v^2}\right]\frac{\partial}{\partial \hat{\theta}_{co}} \\ &+ (\gamma-1)\frac{v_\theta v_\phi}{v^2}\frac{\partial}{\partial \hat{\phi}_{co}}, \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} &= -\gamma v_\phi \frac{\partial}{\partial \hat{t}_{co}} + (\gamma-1)\frac{v_\phi v_r}{v^2}\frac{\partial}{\partial \hat{r}_{co}} \\ &+ (\gamma-1)\frac{v_\phi v_\theta}{v^2}\frac{\partial}{\partial \hat{\theta}_{co}} \\ &+ \left[1 + (\gamma-1)\frac{v_\phi^2}{v^2}\right]\frac{\partial}{\partial \hat{\phi}_{co}}. \end{aligned} \tag{20}$$

### 3.3 Radiation Moments

Radiation moments can be defined from the specific intensity, $I_\nu(x^\alpha;\mathbf{n})$, either in a fixed tetrad or in a comoving tetrad as done in flat spacetime.

The radiation energy density is equal to the zeroth moment

$$E = \iint I_\nu d\nu d\Omega, \quad E_{co} = \iint I_{\nu_{co}} d\nu_{co} d\Omega_{co}, \tag{21}$$

where $d\Omega$ and $d\Omega_{co}$ are the solid angle elements in the fixed and comoving tetrad, respectively. The radiation flux consists of three first moments

$$F^i = \iint I_\nu n^i d\nu d\Omega, \quad F^i_{co} = \iint I_{\nu_{co}} n^i_{co} d\nu_{co} d\Omega_{co}, \tag{22}$$

where $i = r, \theta, \phi$. The radiation pressure tensor is symmetric and consists of six second moments

$$P^{ij} = \iint I_\nu n^i n^j d\nu d\Omega, \quad P^{ij}_{co} = \iint I_{\nu_{co}} n^i_{co} n^j_{co} d\nu_{co} d\Omega_{co}. \tag{23}$$

From the definition of the radiation stress tensor (Eq. 2), the tetrad components of the radiation stress tensor for the fixed tetrad are simply

$$R^{\hat\alpha\hat\beta} = \begin{pmatrix} E & F^r & F^\theta & F^\phi \\ F^r & P^{rr} & P^{r\theta} & P^{r\phi} \\ F^\theta & P^{r\theta} & P^{r\theta} & P^{\theta\phi} \\ F^\phi & P^{r\phi} & P^{\theta\phi} & P^{\phi\phi} \end{pmatrix}. \tag{24}$$

For the spherically symmetric radiation field, $F^\theta = F^\phi = P^{r\theta} = P^{r\phi} = 0$ and $P^{\theta\theta} = P^{\phi\phi} = (E - P^{rr})/2$. The comoving tetrad components are similarly

$$R^{\hat\alpha\hat\beta}_{co} = \begin{pmatrix} E_{co} & F^r_{co} & F^\theta_{co} & F^\phi_{co} \\ F^r_{co} & P^{rr}_{co} & P^{r\theta}_{co} & P^{r\phi}_{co} \\ F^\theta_{co} & P^{r\theta}_{co} & P^{r\theta}_{co} & P^{\theta\phi}_{co} \\ F^\phi_{co} & P^{r\phi}_{co} & P^{\theta\phi}_{co} & P^{\phi\phi}_{co} \end{pmatrix}. \tag{25}$$

The contravariant components of the radiation stress tensor $R^{\alpha\beta}$ are related to the fixed tetrad components by the transformation

$$R^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x^{\hat\mu}} \frac{\partial x^\beta}{\partial x^{\hat\nu}} R^{\hat\mu\hat\nu}, \tag{26}$$

and contain all the curvature and coordinate specifics,

$$R^{\alpha\beta} = \begin{pmatrix} \Gamma^{-2}E & F^r & \Gamma^{-1}\frac{F^\theta}{r} & \Gamma^{-1}\frac{F^\phi}{r\sin\theta} \\ F^r & \Gamma^2 P^{rr} & \Gamma\frac{P^{r\theta}}{r} & \Gamma\frac{P^{r\phi}}{r\sin\theta} \\ \Gamma^{-1}\frac{F^\theta}{r} & \Gamma\frac{P^{r\theta}}{r} & \frac{P^{\theta\theta}}{r^2} & \frac{P^{\theta\phi}}{r^2\sin\theta} \\ \Gamma^{-1}\frac{F^\phi}{r\sin\theta} & \Gamma\frac{P^{r\phi}}{r\sin\theta} & \frac{P^{\theta\phi}}{r^2\sin\theta} & \frac{P^{\phi\phi}}{r^2\sin^2\theta} \end{pmatrix}. \tag{27}$$

Since tetrad components $R^{\hat\alpha\hat\beta}$ and $R^{\hat\alpha\hat\beta}_{co}$ are Lorentz tensors, they are related by the Lorentz boost $\Lambda^{\hat\alpha}{}_{\hat\beta}(\boldsymbol{v})$,

$$R^{\hat\alpha\hat\beta}_{co} = \Lambda^{\hat\alpha}{}_{\hat\lambda}(-\boldsymbol{v})\Lambda^{\hat\beta}{}_{\hat\mu}(-\boldsymbol{v})R^{\hat\lambda\hat\mu}. \tag{28}$$

Substituting equations (24) and (25) yields the standard transformation law between the fixed-frame radiation moments and the comoving-frame radiation moments (Mihalas & Mihalas 1984; Munier & Weaver 1986), which is reproduced here for completeness:

$$E_{co} = \gamma^2\left[E - 2v_iF^i + v_iv_jP^{ij}\right] \tag{29}$$

$$\begin{aligned} F^i_{co} &= -\gamma^2 v^i E + \gamma\left[\delta^i_j + (\frac{\gamma-1}{v^2}+\gamma)v^iv_j\right]F^j \\ &\quad -\gamma v_j\left[\delta^i_k + \frac{\gamma-1}{v^2}v^iv_k\right]P^{jk} \end{aligned} \tag{30}$$

$$\begin{aligned} P^{ij}_{co} &= \gamma^2 v^iv^jE - \gamma\left[v^i\delta^j_k + v^j\delta^i_k + 2\frac{\gamma-1}{v^2}v^iv^jv_k\right]F^k \\ &\quad +(\delta^i_k + \frac{\gamma-1}{v^2}v^iv_k)(\delta^j_k + \frac{\gamma-1}{v^2}v^jv_l)P^{kl}. \end{aligned} \tag{31}$$

Since this transformation is between tetrad quantities, it is valid in curved spacetime as well, as long as the proper velocity $\boldsymbol{v}$ is used. Replacing $\boldsymbol{v}$ with $-\boldsymbol{v}$ yields the inverse transformation. As discussed in detail in Mihalas & Mihalas (1984), the comoving-frame radiation flux can be significantly different from the fixed-frame flux due to the advection of radiation energy density and pressure, $v^iE + v_jP^{ij}$, and one should carefully determine which flux is the appropriate one, especially in a lower-order approximation.

### 3.4 Radiation Four-Force Density

The energy and momentum transfer rates, measured by a comoving observer, are given by the comoving tetrad components of the radiation four-force density,

$$G^{\hat\alpha}_{co} = \frac{1}{c}\int d\nu_{co}\int d\Omega_{co}[\chi_{co}I_{\nu_{co}} - \eta_{co}]n^{\hat\alpha}_{co}. \tag{32}$$

In terms of the usual heating function $\Gamma_{co}$, the cooling function $\Lambda_{co}$ (both per unit proper volume) and the mean opacity $\chi_{co}$ (per unit proper length), all in the comoving frame,

$$G^{\hat t}_{co} = \Gamma_{co} - \Lambda_{co} \tag{33}$$

$$G^{\hat i}_{co} = \bar\chi_{co}F^i_{co}, \tag{34}$$

where

$$\Gamma_{co} \equiv \frac{1}{c}\int d\nu_{co}\int d\Omega_{co}\chi_{co}I_{\nu_{co}}, \tag{35}$$

$$\Lambda_{co} \equiv \frac{1}{c}\int d\nu_{co}\int d\Omega_{co}\eta_{co}, \tag{36}$$

$$\bar\chi_{co}F^i_{co} \equiv \frac{1}{c}\int d\nu_{co}\int d\Omega_{co}\chi_{co}I_{\nu_{co}}n^i_{co}. \tag{37}$$

Expression in the form of equation (34) is possible only if the thermally emitted (and/or scattered) photons are isotropic in the comoving frame so that the net momentum they remove from the gas is zero.

The components $G^\alpha$ are related to the comoving tetrad components $G^{\hat\alpha}_{co}$ by the transformation

$$G^\alpha = \frac{\partial x^\alpha}{\partial x^{\hat\beta}_{co}}G^{\hat\beta}_{co}. \tag{38}$$

Substituting $\partial x^\alpha/\partial x^{\hat\beta}_{co}$ from equation (19) yields

$$\begin{aligned} G^t &= \frac{\gamma}{\Gamma}\left[G^{\hat t}_{co} + v_iG^{\hat i}_{co}\right] \\ G^r &= \Gamma\left[G^{\hat r}_{co} + \gamma v_rG^{\hat t}_{co} + \frac{\gamma-1}{v^2}v_rv_iG^{\hat i}_{co}\right] \\ G^\theta &= \frac{1}{r}\left[G^{\hat\theta}_{co} + \gamma v_\theta G^{\hat t}_{co} + \frac{\gamma-1}{v^2}v_\theta v_iG^{\hat i}_{co}\right] \\ G^\phi &= \frac{1}{r\sin\theta}\left[G^{\hat\phi}_{co} + \gamma v_\phi G^{\hat t}_{co} + \frac{\gamma-1}{v^2}v_\phi v_iG^{\hat i}_{co}\right]. \end{aligned} \tag{39}$$

## 4 RADIATION HYDRODYNAMIC EQUATIONS

### 4.1 Continuity Equation

The continuity equation in the Schwarzschild cordinates (8) is

$$\frac{1}{\Gamma^2}\frac{\partial}{\partial t}(yn) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 nU^r) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta nU^\theta) + \frac{\partial}{\partial\phi}(nU^\phi) = 0, \quad (40)$$

which is the conservation equation of the particle number, rather than the mass density, because the mass density $\rho$ in relativity necessarily contains the internal energy that is not always conserved.

The continuity equation for spherically symmetric, steady-state flow becomes

$$4\pi r^2 nU^r = -\dot{N} = \text{constant}. \quad (41)$$

One should note that the velocity $U^r$ in equation (41) is the radial component of the four-velocity $U^\alpha$ which is different from the proper velocity $v^r$. It is the proper velocity $v^r = y^{-1}U^r$ that is always equal to $c$ at the black hole horizon ($r = 2m$) regardless of the value of $U^r$ while $U^r$ is not necessarily equal to $c$ at the horizon. This applies to non-spherical cases as well.

### 4.2 Hydrodynamic Equations

The projection tensor ${P_\alpha}^\beta$ projects a tensor on a direction perpendicular to $U^\alpha$,

$${P_\alpha}^\beta = {g_\alpha}^\beta + U_\alpha U^\beta = {\delta_\alpha}^\beta + U_\alpha U^\beta. \quad (42)$$

Relativistic Euler equation is obtained by projecting equation (6) with ${P_\alpha}^\beta$ to get ${P_\beta}^\alpha {T^{\beta\lambda}}_{;\lambda} = {P_\beta}^\alpha G^\beta$, which becomes

$$\omega_g {U^\alpha}_{;\beta}U^\beta + g^{\alpha\beta}P_{g,\beta} + U^\alpha U^\beta P_{g,\beta} = G^\alpha + U^\alpha U_\beta G^\beta. \quad (43)$$

Radial part of Euler equation is obtained by fixing $\alpha = r$,

$$\begin{aligned}
& \omega_g U^t\frac{\partial U^r}{\partial t} + \omega_g U^i\frac{\partial U^r}{\partial x^i} \\
+ \; & \omega_g\frac{m}{r^2}\left[\Gamma^2(U^t)^2 - \Gamma^{-2}(U^r)^2\right] \\
- \; & \omega_g\frac{\Gamma^2}{r}\left[(rU^\theta)^2 + (r\sin\theta U^\phi)^2\right] \\
+ \; & U^r U^t\frac{\partial P_g}{\partial t} + \Gamma^2\frac{\partial P_g}{\partial r} + U^r U^i\frac{\partial P_g}{\partial x^i} \\
= \; & -yU^r G^t + [1+\Gamma^{-2}(U^r)^2]G^r + r^2 U^r U^\theta G^\theta \\
& + r^2\sin^2\theta U^r U^\phi G^\phi. \qquad (44)
\end{aligned}$$

The term $mr^{-2}$ on the left-hand side represents the gravitational acceleration and $\Gamma^2 r^{-1}[(rU^\theta)^2 + (r\sin\theta U^\phi)^2]$ the centrifugal acceleration. The right-hand side of the equation consists of all four components of $G^\alpha$, not just $G^r$. In a spherically symmetric case, the equation becomes (Park 1993)

$$\begin{aligned}
\frac{y}{\Gamma}\frac{\partial U^r}{\partial t} \; & + \; \frac{1}{2}\frac{\partial (U^r)^2}{\partial r} + \frac{m}{r^2} + \frac{y}{\Gamma}\frac{U^r}{\omega_g}\frac{\partial P_g}{\partial t} + \frac{y^2}{\omega_g}\frac{\partial P_g}{\partial r} \\
& = \; \frac{y}{\omega_g}G^{\hat{r}}_{co} = \frac{y}{\omega_g}\bar{\chi}_{co}F^r_{co}, \qquad (45)
\end{aligned}$$

and the comoving-frame flux $F^r_{co}$, rather than the fixed-frame flux $F^r$, is directly responsible for the radial acceleration. Therefore, even when the net flux $F^r$ measured by an observer at rest with respect to the coordinate is small, gas particles can still experience significant radiative force in the presence of significant radiation energy density $E$ and pressure $P^{ij}$ in the optically thick medium, and slow down efficiently (Park & Miller 1991).

The $\theta$-part of the Euler equation is

$$\begin{aligned}
& \omega_g U^t\frac{\partial U^\theta}{\partial t} + \omega_g U^i\frac{\partial U^\theta}{\partial x^i} \\
+ \; & 2\omega_g\frac{1}{r}U^r U^\theta - \omega_g\sin\theta\cos\theta(U^\phi)^2 \\
+ \; & U^\theta U^t\frac{\partial P_g}{\partial t} + \frac{1}{r^2}\frac{\partial P_g}{\partial\theta} + U^\theta U^i\frac{\partial P_g}{\partial x^i} \\
= \; & -yU^\theta G^t + \Gamma^{-2}U^r U^\theta G^r + [1+r^2(U^\theta)^2]G^\theta \\
& + r^2\sin^2\theta U^\theta U^\phi G^\phi, \qquad (46)
\end{aligned}$$

and the $\phi$-part

$$\begin{aligned}
& \omega_g U^t\frac{\partial U^\phi}{\partial t} + \omega_g U^i\frac{\partial U^\phi}{\partial x^i} \\
+ \; & 2\omega_g\frac{1}{r}U^r U^\phi + 2\omega_g\cot\theta U^\theta U^\phi \\
+ \; & U^\phi U^t\frac{\partial P_g}{\partial t} + \frac{1}{r^2\sin^2\theta}\frac{\partial P_g}{\partial\phi} + U^\phi U^i\frac{\partial P_g}{\partial x^i} \\
= \; & -yU^\phi G^t + \Gamma^{-2}U^r U^\phi G^r + r^2 U^\theta U^\phi G^\theta \\
& + [1+r^2\sin^2\theta(U^\phi)^2]G^\phi. \qquad (47)
\end{aligned}$$

Energy equation is more readily obtained by projecting equation (6) along the four-velocity $U_\alpha {T^{\alpha\beta}}_{;\beta} = U_\alpha G^\alpha$,

$$\begin{aligned}
& -nU^t\frac{\partial}{\partial t}\left(\frac{\omega_g}{n}\right) - nU^i\frac{\partial}{\partial x^i}\left(\frac{\omega_g}{n}\right) + U^t\frac{\partial P_g}{\partial t} + U^i\frac{\partial P_g}{\partial x^i} \\
= \; & -yG^t + \Gamma^{-2}U^r G^r + r^2 U^\theta G^\theta + r^2\sin^2\theta U^\phi G^\phi \qquad (48) \\
= \; & -G^{\hat{t}}_{co} = \Lambda_{co} - \Gamma_{co}.
\end{aligned}$$

Again, the exchange rate of energy between matter and radiation is expressed by heating and cooling functions measured in a comoving frame.

### 4.3 Radiation Moment Equations

The zeroth moment equation, i.e., the radiation energy equation, is obtained by taking the time component ($\alpha = t$) in equation (7) and expressing $G^\alpha$ with the corresponding radiation moments,

$$\begin{aligned}
& \frac{1}{\Gamma^2}\frac{\partial E}{\partial t} + \frac{1}{\Gamma^2 r^2}\frac{\partial}{\partial r}(r^2\Gamma^2 F^r) \\
+ \; & \frac{1}{\Gamma r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F^\theta) + \frac{1}{\Gamma r\sin\theta}\frac{\partial}{\partial\phi}(F^\phi) \qquad (49) \\
= \; & -G^t = -\frac{y}{\Gamma^2}\left(\Gamma_{co} - \Lambda_{co} + \bar{\chi}_{co}v_i F^i_{co}\right).
\end{aligned}$$

In equation (49), both the temporal and spatial derivatives are applied to the fixed-frame radiation moments while the interaction between matter and radiation is described by the comoving frame quantities. This 'mixed-frame' representation simplifies the equation and makes each term easier to understand. For example, $\Lambda_{co} - \Gamma_{co}$ is the energy input from the matter to the radiation field through the usual radiative heating and cooling processes and $\bar{\chi}_{co}v_i F^i_{co}$ is the work

done by the photons to the matter via transfer of momentum through scattering or absorption (Park & Miller 1991).

When the spherically symmetric radiation field does not vary in time nor interacts with matter, equation (49) becomes the 'generalized luminosity conservation' equation,

$$4\pi r^2 \Gamma^2 F^r = L_\infty, \tag{50}$$

where the constant $L_\infty$ is the luminosity measured by an observer at infinity. A static observer at a given radius $r$ finds the luminosity to be a function of radius,

$$L_r \equiv 4\pi r^2 F^r = \frac{L_\infty}{1-2m/r}. \tag{51}$$

This is the well-known gravitational redshift effect: locally measured (within the gravitational potential) fixed-frame luminosity can be larger than the luminosity seen by an observer far away (outside the gravitational potential). For example, the fixed-frame luminosity near the compact object can be super-Eddington even when the luminosity measured far away is sub-Eddington (Park 1992).

The $r$-part of equation (7) yields the radiation momentum equation in $r$-direction,

$$\begin{aligned} & \frac{\partial F^r}{\partial t} + \Gamma^2 \frac{\partial P^{rr}}{\partial r} + \frac{\Gamma}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta P^{r\theta}\right) + \frac{\Gamma}{r\sin\theta}\frac{\partial P^{r\phi}}{\partial\phi} \\ + \; & \frac{m}{r^2}\left(E + P^{rr}\right) + \frac{\Gamma^2}{r}\left(2P^{rr} - P^{\theta\theta} - P^{\phi\phi}\right) \\ = \; & -G^r \\ = \; & -\Gamma\bar{\chi}_{co}F^r_{co} - \Gamma\gamma v_r(\Gamma_{co} - \Lambda_{co}) - \Gamma\frac{\gamma-1}{v^2}v_r v_i \bar{\chi}_{co}F^i_{co}. \end{aligned} \tag{52}$$

In relativistic treatment, $v^1$-order term $v_r(\Gamma_{co} - \Lambda_{co})$ and the redshift term $(m/r)(E+P)/r$ should be included as well as the correction factors $\Gamma$ and $y$. These relativistic terms are negligible in slowly moving, optically thin flow. However, when $\tau v \gtrsim 1$, they significantly affect the radiation field as well as the dynamics of the gas flow (Park 1990; Park & Miller 1991; Park 1992; Park & Miller 1995).

The radiation momentum equation in $\theta$-direction from equation (7) is

$$\begin{aligned} & \frac{1}{\Gamma}\frac{\partial F^\theta}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r P^{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta P^{\theta\theta}\right) \\ + \; & \frac{1}{r\sin\theta}\frac{\partial P^{\theta\phi}}{\partial\phi} + \frac{2\Gamma}{r}P^{r\theta} - \frac{1}{r\tan\theta}P^{\phi\phi} \\ = \; & -rG^\theta \\ = \; & -\bar{\chi}_{co}F^\theta_{co} - \gamma v_\theta(\Gamma_{co} - \Lambda_{co}) - \frac{\gamma-1}{v^2}v_\theta v_i \bar{\chi}_{co}F^i_{co}, \end{aligned} \tag{53}$$

and that in $\phi$-direction is

$$\begin{aligned} & \frac{1}{\Gamma}\frac{\partial F^\phi}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\Gamma r P^{r\phi}\right) + \frac{1}{r}\frac{\partial P^{\theta\phi}}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial P^{\phi\phi}}{\partial\phi} \\ + \; & \frac{2\Gamma}{r}P^{r\phi} + \frac{2}{r\tan\theta}P^{\theta\phi} \\ = \; & -r\sin\theta G^\phi \\ = \; & -\bar{\chi}_{co}F^\phi_{co} - \gamma v_\phi(\Gamma_{co} - \Lambda_{co}) - \frac{\gamma-1}{v^2}v_\phi v_i \bar{\chi}_{co}F^i_{co}. \end{aligned} \tag{54}$$

In a static non-relativistic case, equations (52–54) reduce to the familiar diffusion equation

$$F^i = -\frac{1}{\bar{\chi}_{co}}P^{ij}{}_{;j}. \tag{55}$$

### 4.4 Discussion

Although reducing the angle-dependent radiative transfer equation to the radiation moment equations lowers the dimensionality of the problem from seven (space, time, angle, and frequency) to four (space and time), it comes with a price: while the number of moment equations (49) and (52-54) is only four, the number of all radiation moments to be determined, $E$, $F^i$, and $P^{ij}$, is ten. This is a generic problem with moment equations: equations at each order always contain successively higher order moments. Astronomers routinely employ some kind of closure relation to close the equations, for example, the Eddington factors. In complex radiation hydrodynamic calculations, an educated guess of the Edddington factors as functions of the optical depth is sometimes tried (see, e.g., Tamazawa et al. 1975, Park 1990).

But it can't be stressed enough that the correct form of the Eddington factors or any other closure relations can only be obtained by solving the full angle-dependent radiative transfer equation (see e.g., Auer & Mihalas 1970, Hummer & Rybicki 1971, Yin & Miller 1995), which has been rarely tried for three-dimensional flows and much less for relativistic (Schinder & Bludman 1989) or time-dependent flows . Even in one-dimensional systems, the viable method, iterative variable Eddington method, requires expensive computations and, moreover, when the flow motion is not monotonic various angle-frequency components couple along a ray and even posing boundary conditions become a problem (Mihalas 1980; Nobili, Turolla, & Zampieri 1993). Iteration method by Schmid-Burgk (1978) seems to be the only workable approach for the general relativistic system, but it works only for spherically symmetric cases. Although there had been a suggestion for the general shape of the Eddginton factors for arbitrary three-dimensional radiation hydrodynamic systems (Minerbo 1978), its validity has yet to be verified by true angle-dependent radiative transfer calculations for a variety of cases (Schinder & Bludman 1989). Needless to say, in special cases where the specific intensity or the radiation moments can be directly calculated, the incomplete radiation moment equations can be eschewed altogether (e.g., Chattopadhyay 2005).

Other related problem with the radiation moment formalism is that the exact nature of the radiation field can be properly described only by the infinite number of moments. Using a finite number of moments by terminating higher order moments or assuming some kind of closure relation can be a good enough approximation in certain conditions, but not necessarily so in any situation: it can lead to spurious pathological behaviours such as a radiation shock, especially when the velocity gradient is steep (Turolla & Nobili 1988; Dullemond 1999).

## 5 SUMMARY

General relativistic, radiation hydrodynamic equations are derived using a covariant tensor formulation. Matter and radiation quantities are defined in the fixed and comoving tetrads and transformed to corresponding covariant forms. The interactions between matter and radiation are described by the radiation four-force density defined in the comov-

ing tetrad and transformed to the covariant form. Conservation of energy and momentum separately for matter and radiation yield the hydrodynamic and radiation moment equations. Current approach is applicable to any three-dimensional systems in any spacetime or coordinates, but in this work, we show explicitly how the equations are derived for the Schwarzschild spacetime. Certain aspects of relativistic radiation hydrodynamics and the fundamental limitation of the radiation moment formalism have been discussed as well.

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## REFERENCES

Auer L. H., Mihala D., 1970, MNRAS, 149, 65
Begelman M. C., 1978, MNRAS, 184, 53
Castor J. I., 1972, ApJ, 178, 779
Chattopadhyay I., 2005, MNRAS, 356, 145
Dullemond C. P., 1999, A&A, 343, 1030
Flammang, R. A., 1982, MNRAS, 199, 833
Flammang, R. A., 1984, MNRAS, 206, 589
Hauschildt P. H., Wehrse R., 1991, JQSRT, 46, 81
Hummer D. G., Rybicki G. B., 1971, MNRAS, 152, 1
Liebendörfer M., Messer O. E. B., Mezzacappa A., Bruenn S. W., Cardall C. Y., Thielemann F.-K., 2004, ApJS, 150, 263
Lindquist R. W., 1966, Ann. Phys., 37, 487
Mihalas D., 1980, ApJ, 237, 574
Mihalas D. , Mihalas B. W., 1984, Foundations of Radiation Hydrodynamics, Oxford Univ. Press, Oxford
Miller G. S., 1990, ApJ, 356, 572
Minerbo G. N., 1978, J. Quant. Spectosc. Radiat. Transfer, 20, 541
Munier A., Weaver R., 1986, Comp. Phys. Rep., 3, 125
Nobili L., Turolla R., Zampieri L., 1991, ApJ, 383, 250
Nobili L., Turolla R., Zampieri L., 1993, ApJ, 404, 686
Park M.-G., 1990, ApJ, 354, 64
Park M.-G., 1992, J. Kor. Phys. Soc., 25, S294
Park M.-G., 1993, A&A, 274, 642
Park M.-G., 2004, J. of Korean Phys. Soc., 45, 1531
Park M.-G., Miller, G. S., 1991, ApJ, 371, 708
Park M.-G., Miller, G. S., 1995, J. Kor. Phys. Soc., 28, S555
Schinder P. J., 1988, PhRvD, 38, 1673
Schinder P. J., Bludman S. A., 1989, ApJ, 346, 350
Schmid-Burgk J., 1978, Ap&SS, 56, 191
Service A. T., 1986, ApJ, 307, 60
Tamazawa S., Toyama K., Kaneko N., Ono Y., 1975, Ap&SS, 32, 403
Thomas, L. H., 1930, Quart. J. Math., 1, 239
Thorne K. S., 1981, MNRAS, 194, 439
Turolla R., Nobili L., 1988, MNRAS, 235, 1273
Yin W.-W., Miller, G. S., 1995, ApJ, 449, 826
Zampieri L., Turolla R., Treves A., 1993, ApJ, 419, 311
Zane S., Turolla R., Nobili L., Erna M., 1996, ApJ, 466, 871

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